1) $X_n : \Omega \longrightarrow [0,1] \cap \mathbb{Z}$

X_5 v.a. binomiale (conta il numero di successi) $p = \frac{1}{3}$ e $n = 5$

$X_k \sim B(5, 1/3)$ e $p(k) = P(X_n = k) = p(T)^k p(C)^{n-k} \binom{n}{k}$

$F(x) = P(X \leq x) = \sum_{k \leq x} p(k)$

$p(0) = \left(\frac{2}{3}\right)^5 \binom{5}{0} = \frac{32}{243} \sim 0.13$ $\quad p(1) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^4 \binom{5}{1} = \frac{80}{243} \sim 0.33$

$p(2) = \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^3 \binom{5}{2} = \frac{80}{243} \sim 0.33$ $\quad p(3) = \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^2 \binom{5}{3} = \frac{40}{243} \sim 0.16$

$p(4) = \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right) \binom{5}{4} = \frac{10}{243} \sim 0.04$ $\quad p(5) = \left(\frac{1}{3}\right)^5 \binom{5}{5} = \frac{1}{243} \sim 0.004$

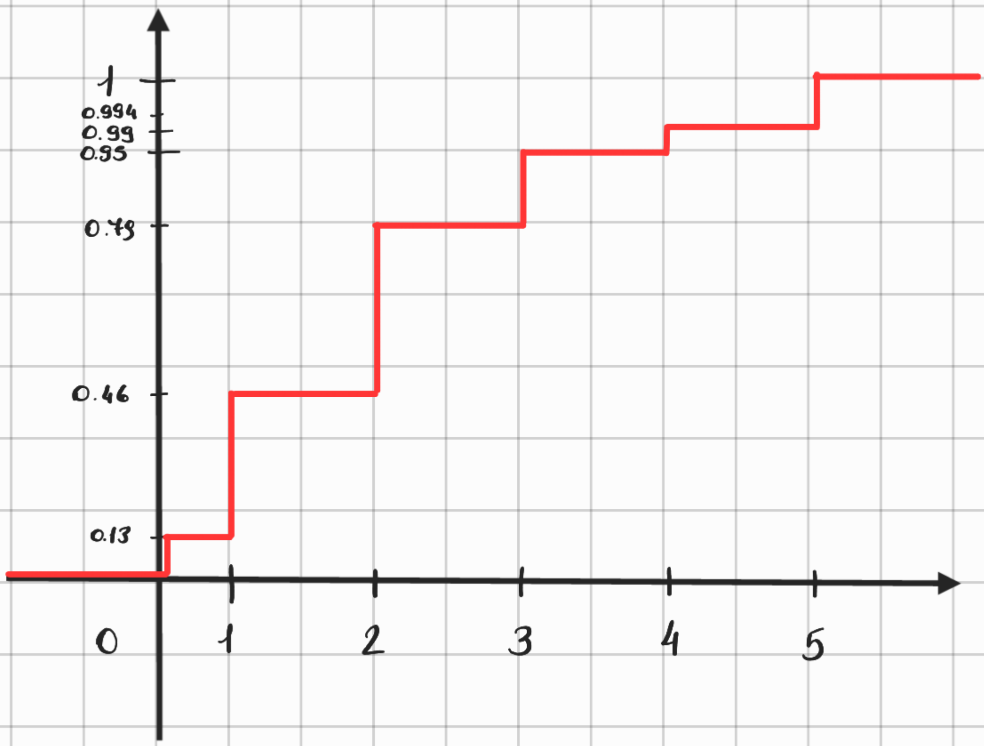
$F(0) = p(0) = 0.13$ $\quad F(1) = p(0) + p(1) = 0.13 + 0.33 = 0.46$

$F(2) = F(1) + p(2) = 0.46 + 0.33 = 0.79$

$F(3) = F(2) + p(3) = 0.79 + 0.16 = 0.95$

$F(4) = F(3) + p(4) = 0.95 + 0.04 = 0.99$

$F(5) = F(4) + p(5) = 0.99 + 0.004 = 0.994$



(i) 0.96 quantile è il numero x t.c. $P(X \leq x) \geq 0.96$ e $P(X \geq x) \geq 0.04$

$P(X \leq 4) = 0.13 + 0.33 + 0.33 + 0.16 + 0.04 = 0.99 \geq 0.96$ ✓

$P(X \geq 4) = 0.04 + 0.004 = 0.044 \geq 0.04$ ✓

4 è lo 0.96 quantile

(ii)

$P(X \geq 2.5) = 0.16 + 0.04 + 0.004 = 0.204 \sim \frac{17}{81}$ ✓

2) $Y_n : \Omega \longrightarrow [n, 2n] \cap \mathbb{Z}$ e $Y_5 \sim B(5, 1/3)$

$p(k) = P(X_n = k) = p(T)^k \, p(C)^{n-k} \binom{n}{k}$

$p(5) = \left(\frac{2}{3}\right)^5 \binom{5}{0} = \frac{32}{243} \sim 0.13$ $\quad p(6) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^4 \binom{5}{1} = \frac{80}{243} \sim 0.33$

$p(7) = \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^3 \binom{5}{2} = \frac{80}{243} \sim 0.33$ $\quad p(8) = \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^2 \binom{5}{3} = \frac{40}{243} \sim 0.16$

$p(9) = \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right) \binom{5}{4} = \frac{10}{243} \sim 0.04$ $\quad p(10) = \left(\frac{1}{3}\right)^5 \binom{5}{5} = \frac{1}{243} \sim 0.004$

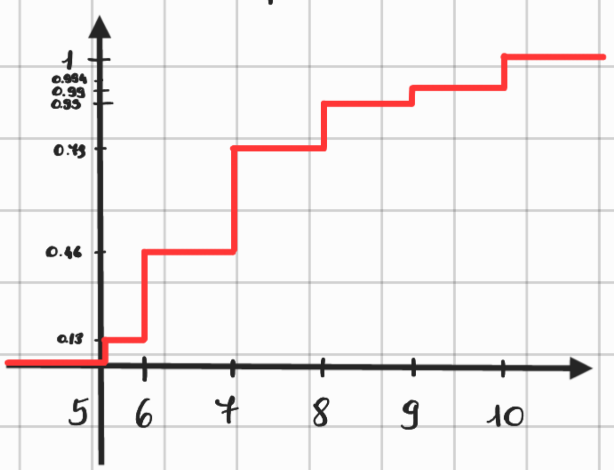
$F(0) = p(0) = 0.13$ $\quad F(1) = p(0) + p(1) = 0.13 + 0.33 = 0.46$

$F(2) = F(1) + p(2) = 0.46 + 0.33 = 0.79$

$F(3) = F(2) + p(3) = 0.79 + 0.16 = 0.95$

$F(4) = F(3) + p(4) = 0.95 + 0.04 = 0.99$

$F(5) = F(4) + p(5) = 0.99 + 0.004 = 0.994$



(i)

$P(Y_5 \le x) \geq 0.5$ e $P(Y_5 \geq x) \geq 0.5$

$P(Y_5 \le 7) = 0.13 + 0.33 + 0.33 = 0.79 \geq 0.5$ ✓

$P(Y_5 \geq 7) = 0.33 + 0.16 + 0.04 + 0.004 = 0.534 \geq 0.5$ ✓

7 é la mediana

3) $X_n : \Omega \to [-n, 2n] \cap \mathbb{Z}$ $\quad X_3 \sim B(3, 1/2)$

$p(k) = P(X_3 = k) = \left(\frac{1}{2}\right)^n \binom{n}{\#\text{successi}}$

$p(-3) = \left(\frac{1}{2}\right)^3 \binom{3}{0} = \frac{1}{8} = p(6)$

$p(-2) = p(5) = p(-1) = p(1) = p(2) = p(4) = 0$

$p(0) = p(3) = \left(\frac{1}{2}\right)^3 \binom{3}{1} = \frac{1}{8} \cdot 3 = 3/8$

4) 3R 5B $\qquad$ 4R 4B

S1 $\qquad$ S2

$A_1 = \{\text{scelgo S1}\}$ $\quad A_2 = \{\text{scelgo S2}\}$ $\qquad P(A_1) = P(A_2) = 1/2$

$X : \Omega \to [0, 3]$ e $X \sim B(3, 1/2)$

$P(X = k) = P(X = k \mid A_1) P(A_1) + P(X = k \mid A_2) P(A_2)$

$P(X = k \mid A_1) = \left(\frac{3}{8}\right)^k \left(\frac{5}{8}\right)^{3-k} \binom{3}{k}$ e $P(X = k \mid A_2) = \left(\frac{1}{2}\right)^3 \binom{3}{k}$

$P(X = 2) = \left(\frac{3}{8}\right)^2 \left(\frac{5}{8}\right) \cdot 3 \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^3 3 \cdot \frac{1}{2} = \frac{135}{2 \cdot 8^3} + \frac{3}{2 \cdot 8} = \frac{327}{1024}$

$$P(A_1 \mid X=2) = \frac{P(X=2 \mid A_1)\,P(A_1)}{P(X=2)} = \frac{135}{\cancel{1024}} \cdot \frac{\cancel{1024}}{327} = \frac{45}{109}$$

5) $p(k) = \left(1-\frac{1}{2}\right)^{k-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^{k-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^{k}$

0.9 quantile è r t.c. $P(X \leq r) \geq 0.9$ e $P(X \geq r) \geq 0.1$

$p(1) = \frac{1}{2}$ $\quad p(2) = \frac{1}{4}$ $\quad p(3) = \frac{1}{8}$ $\quad p(4) = \frac{1}{16}$

$P(X \leq 4) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} = 0.9375 \geq 0.9$

$P(X \geq 4) = \sum_{i=4}^{\infty} \frac{1}{2^i} = 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} = \frac{3}{8} \geq 0.1$

4 è lo 0.9 quantile

6) $p(k) = P(X=k) = e^{-3} \cdot \frac{3^k}{k!}$

$p(0) = e^{-3} \cdot \frac{3^0}{0!} = \frac{1}{e^3} \sim 0.05$ $\qquad p(1) = e^{-3} \cdot 3 = \frac{3}{e^3} \sim 0.15$

$p(2) = e^{-3} \cdot \frac{3^2}{2} = \frac{9}{2e^3} \sim 0.225$

$P(X \leq 2) = 0.05 + 0.15 + 0.225 = 0.38 \geq 0.25$

$P(X \geq 2) = e^{-3} \sum_{i=2}^{\infty} \frac{3^i}{i!} \sim e^{-3}(e^3 - 1 - 3) \sim 0.8 \geq 0.75$

2 è il primo quartile

7)
$$F(x) = \begin{cases} 0 & x < 0 \\ \int_0^x \frac{1}{9}\,dt = [t/9]_0^x = \frac{x}{9} & 0 \leq x \leq 3 \\ \int_0^3 \frac{1}{9}\,dt + \int_3^x \frac{2}{9}\,dt = \left[\frac{1}{3} + \frac{2t}{9}\right]_3^x = \frac{1}{3} + \frac{2x}{9} - \frac{6}{9} = \frac{2x-3}{9} & 3 < x \leq 6 \\ 1 & x > 6 \end{cases}$$

(i)

$$P(X \in A) = P(1 \leq X \leq 5) = P(X \leq 5) - P(X \leq 1) = F(5) - F(1) = \frac{7}{9} - \frac{1}{9} = \frac{6}{9}$$

(ii)

$$P(X \in A \mid X \in B) = \frac{P(X \in [1,4])}{P(X \in [-1,4])} = \frac{F(4) - F(1)}{F(4) - F(-1)} = \frac{4}{\cancel{9}} \cdot \frac{\cancel{9}}{5} = \frac{4}{5}$$

(iii)

$0 \leq x \leq 3$ perché $F(3) = \frac{1}{3} > 0.25$ $\quad F(x) = 0.25$

$\frac{x}{9} = \frac{1}{4} \rightarrow x = \frac{9}{4}$ primo quartile

$3 < x \leq 6$ perché $F(7) = 1 > 0.5$ $\quad F(x) = 0.5$

$\frac{2x-3}{9} = \frac{1}{2} \longrightarrow x = \left(\frac{1}{2} + \frac{1}{3}\right)\frac{9}{2} = \frac{5}{\cancel{6}_2} \cdot \frac{\cancel{9}^3}{2} = \frac{15}{4}$ mediana

$3 < x \leq 6$ perché $F(7) = 1 > 0.75$ $\quad F(x) = 0.75$

$\frac{2x-3}{9} = \frac{3}{4} \longrightarrow x = \left(\frac{3}{4} + \frac{1}{3}\right)\frac{9}{2} = \frac{13}{12} \cdot \frac{9}{2} = \frac{39}{8}$ terzo quartile

8

(i) Deve valere che $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

$$F(x) = \begin{cases} 0 & x \leq -3 \\ \int_{-3}^{-1} 2\lambda\,dx = [2\lambda x]_{-3}^{-1} = -2\lambda + 6\lambda = 4\lambda & -3 < x < -1 \\ 0 & -1 \leq x \leq 0 \\ 4\lambda + \int_0^{+\infty} \frac{1}{2}\lambda e^{-\lambda x}\,dx = 4\lambda - \frac{1}{2}\int_0^{+\infty} -\lambda e^{-\lambda x}\,dx = 4\lambda + \left[-\frac{1}{2}e^{-\lambda x}\right]_0^{+\infty} = & \end{cases}$$

$$= 4\lambda + \left[-\frac{1}{2} \cdot 0 + \frac{1}{2}\right] \Rightarrow 4\lambda + \frac{1}{2} = 1 \Rightarrow \lambda = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

(ii)

$$P(X \in [-2,2]) = F(2) - F(-2) = \left(4 \cdot \frac{1}{8} + \frac{1}{2} - \frac{1}{2} e^{-\frac{1}{8}\cdot 2}\right) - \left(2 \cdot \frac{1}{8}(-2) + 6 \cdot \frac{1}{8}\right) =$$
$$= \frac{3}{4} - \frac{1}{2} e^{-1/4}$$

(iii)

$F(x) = 0.25 = \frac{1}{4}$ $\quad -3 < x < -1$

9) $X \sim N(1,4) \Rightarrow X = \sigma Z + m$ e $Z \sim N(0,1)$

$X = 2Z + 1$

(i)

$$P(X \in [0,2]) = P(0 \leq 2Z+1 \leq 2) = P(-\tfrac{1}{2} \leq Z \leq \tfrac{1}{2}) = \Phi(\tfrac{1}{2}) - \Phi(-\tfrac{1}{2}) =$$
$$= 2\Phi(\tfrac{1}{2}) - 1 \sim 0.38292$$

(ii)

Dato che $X = 2Z + 1$ allora $P(X \leq x) = P(2Z+1 \leq x) = P(Z \leq \frac{x-1}{2})$ e $P(X \geq x) = P(Z \geq \frac{x-1}{2})$ quindi:

$\Phi(\frac{x-1}{2}) = 0.25 = 1/4$ ma la tavola parte da 0.5 quindi

$1 - \Phi(\frac{x-1}{2}) = 1 - 0.25 \Longrightarrow \Phi(\frac{1-x}{2}) = 0.75$ quindi $\frac{1-x}{2} \sim 0.675$

$x \sim 1 - (2 \cdot 0.675) = -0.35$ primo quartile

$\Phi(\frac{x-1}{2}) = 0.5$ quindi $\frac{x-1}{2} = 0 \Longrightarrow x = 1$ secondo quartile

$\Phi\left(\frac{x-1}{2}\right) = 0.75$ quindi $\frac{x-1}{2} \sim 0.675 \Rightarrow x \sim 1 + (2 \cdot 0.675) = 2.35$

terzo quartile

10) $p(K) = p^K (1-p)^{2-K} \binom{2}{K}$

$p(0) = (1-p)^2 \quad p(1) = p(1-p)2 \quad p(2) = p^2$

$p(X=0, Y=2) = p(0) = (1-p)^2$
$p(X=1, Y=1) = p(1) = p(1-p)2$
$p(X=2, Y=0) = p(2) = p^2$
$p(X_i, Y_i) = 0 \quad \forall$ altro valore

11) $p(K) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 \end{cases}$

Uso la formula di convoluzione

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)\, dx = \int_{-\infty}^{+\infty} f_Y(y) f_X(z-y)\, dy$$

(i)

$$f_{Y_1}(y) = \int_{-\infty}^{+\infty} f_{X_1}(x) f_{X_2}(y-x)\, dx = \int_0^y 2e^{-2x} \cdot 2e^{-2(y-x)} dx =$$

$$= \int_0^y 2\cancel{e^{-2x}} \cdot 2e^{-2y} \cdot \cancel{e^{2x}}\, dx = \left[4e^{-2y} x\right]_0^y = 4ye^{-2y}$$

(ii)

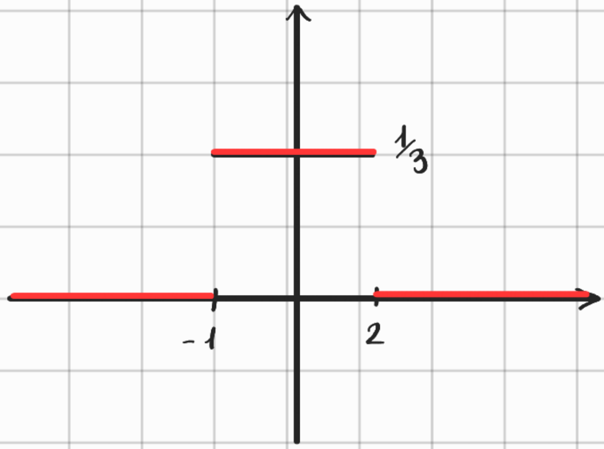
$$P(Y_2 \in [0,1]) \Rightarrow f_{Y_2}(y) = \int_{-\infty}^{+\infty} f_{Y_1}(y) f_{X_3}(x-y)\, dy = \int_0^x 4ye^{-2y} \cdot 2e^{-2(x-y)} dy =$$

$$= \int_0^x 4y\cancel{e^{-2y}} \cdot 2\cancel{e^{2y}} \cdot e^{-2x} dy = \int_0^x 8ye^{-2x} dy = {}^4\cancel{8}e^{-2x} \cdot \frac{x^2}{\cancel{2}} = 4x^2e^{-2x}$$

$$\int_0^1 4x^2e^{-2x} dx = 4\int_0^1 x^2e^{-2x} dx =$$

12)

$$f_X(x) = \begin{cases} \frac{1}{3} & -1 \le x \le 2 \\ 0 \end{cases}$$



$Y_1 = h(X)$ e $h(X) = X^2$

X^2 assume valori in $[0, 4]$

$$F_{Y_1}(y) = P(Y_1 \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

$$F_{Y_1}(y) = \begin{cases} P(-\sqrt{y} \le X \le \sqrt{y}) & y \ge 0 \\ 0 & y < 0 \end{cases} \qquad P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\, dx$$

- $0 \le y \le 1 \longrightarrow \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{3}\, dx = \frac{1}{3}\int_{-\sqrt{y}}^{\sqrt{y}} 1\, dx = \left[\frac{1}{3}x\right]_{-\sqrt{y}}^{\sqrt{y}} = \frac{1}{3}\sqrt{y} + \frac{1}{3}\sqrt{y} = \frac{2}{3}\sqrt{y}$
- $1 < y \le 4 \longrightarrow \int_{-1}^{\sqrt{y}} \frac{1}{3}\, dx = \left[\frac{1}{3}x\right]_{-1}^{\sqrt{y}} = \frac{1}{3}\sqrt{y} + \frac{1}{3} = \frac{1}{3}(\sqrt{y}+1)$
- $y > 4 \longrightarrow \int_{-1}^{2} \frac{1}{3}\, dx = \left[\frac{1}{3}x\right]_{-1}^{2} = \frac{2}{3} + \frac{1}{3} = 1$

$F_{Y_1}(0) = 0 \quad F_{Y_1}(1) = \frac{2}{3} \quad F_{Y_1}(2) = \frac{\sqrt{2}+1}{3} \quad F_{Y_1}(3) = \frac{\sqrt{3}+1}{3} \quad F_{Y_1}(4) = 1$

$$\left.\begin{array}{l} r_{0.25} \in (0,1) \text{ t.c. } \frac{2}{3}\sqrt{r_{0.25}} = \frac{1}{4} \longrightarrow r_{0.25} = \frac{9}{64} \\ r_{0.5} \in (0,1) \text{ t.c. } \frac{2}{3}\sqrt{r_{0.25}} = \frac{1}{2} \longrightarrow r_{0.5} = \frac{9}{16} \\ r_{0.75} \in (1,4) \text{ t.c. } \frac{1}{3}(\sqrt{r_{0.75}}+1) = \frac{3}{4} \longrightarrow r_{0.75} = \frac{25}{16} \end{array}\right\} F(r) = \beta$$

$Y_2 = h(g(x))$ con $g(X) = X+1$ e $h(x) = x^2$ $\quad$ Y_2 assume valori in $[0, 9]$

$h^{-1}(y) = \sqrt{y}$

$$f_{X+1}(x) = f_X(\underbrace{g^{-1}(x)}_{x-1})|1| = f_X(x-1)$$

$$f_{Y_2}(y) = \begin{cases} f_{X+1}(h^{-1}(y)) \left| \frac{d h^{-1}(y)}{dy} \right| \\ 0 \end{cases} = \begin{cases} f_X(h^{-1}(y)-1) \left| \frac{d h^{-1}(y)}{dy} \right| \\ 0 \end{cases} = \frac{f_X(\sqrt{y}-1)}{2\sqrt{y}} = \frac{1}{6\sqrt{y}}$$

$$F_{Y_2}(y) = \begin{cases} 0 & y<0 \\ \int_0^y \frac{1}{6\sqrt{t}}\,dt = \frac{1}{3}\int_0^y \frac{1}{2\sqrt{t}}\,dt = \frac{1}{3}\left[\sqrt{t}\right]_0^y = \frac{1}{3}\sqrt{y} & y \in [0,9] \\ 1 & y>9 \end{cases}$$

$F_{Y_2}(0) = 0$ $\quad F_{Y_2}(1) = \frac{1}{3}$ $\quad F_{Y_2}(2) = \frac{\sqrt{2}}{3}$ $\quad F_{Y_2}(3) = \frac{\sqrt{3}}{3}$ $\quad F_{Y_2}(4) = \frac{2}{3}$

$F_{Y_2}(i) = \frac{\sqrt{i}}{3}$ $\quad i = 5, \dots, 8$ $\quad F_{Y_2}(9) = 1$

$$\frac{1}{3}\sqrt{r_{0.25}} = \frac{1}{4} \longrightarrow r_{0.25} = \frac{9}{16}$$

$$\frac{1}{3}\sqrt{r_{0.5}} = \frac{1}{2} \longrightarrow r_{0.5} = \frac{9}{4}$$

$$\frac{1}{3}\sqrt{r_{0.75}} = 3/4 \longrightarrow r_{0.75} = \frac{81}{16}$$

13) $Z = X \cdot Y$ assume valori in $\{1,2,3,4,6,9\}$

$$P(Z=1) = P(X=1, Y=1) = P(X=1) \cdot P(Y=1) = \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}$$
$$P(Z=2) = P(X=1, Y=2) + P(X=2, Y=1) = \left(\frac{1}{3}\right)^2 + \frac{1}{3} \cdot \frac{1}{6} = \frac{3}{18}$$
$$P(Z=3) = P(X=1, Y=3) + P(X=3, Y=1) = \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{6} = \frac{4}{18}$$
$$P(Z=4) = P(X=2, Y=2) = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$
$$P(Z=6) = P(X=2, Y=3) + P(X=3, Y=2) = \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{6} + \frac{1}{9} = \cancel{15}^{5}/\cancel{54}_{18}$$
$$P(Z=9) = P(X=3, Y=3) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

$$F(x) = \begin{cases} 0 & x < 1 \\ 1/18 & 1 \leq x \leq 2 \\ 2/9 & 2 < x \leq 3 \\ 4/9 & 3 < x \leq 4 \\ 5/9 & 4 < x \leq 6 \\ 45/54 & 6 < x \leq 9 \\ 1 & x > 9 \end{cases}$$

$F(X \leq 4) = \frac{5}{9} = 0.55 \geq 0.5$

e

$F(X \geq 4) \geq \frac{5}{9} \geq 0.5$

(14) $Y = Z + 1$ perché $Y = \sigma Z + m$ con $Z \sim N(0,1)$

$$P(X+Y>0) = P(X=1, 1+Y>0) + P(X=-1, -1+Y>0) =$$
$$= P(X=1) \cdot P(Y>-1) + P(X=-1) P(Y>1) =$$
$$= p \cdot P(Y>-1) + (1-p) P(Y>1) =$$

$Y = Z+1$

$$= p \cdot P(Z>-2) + (1-p) P(Z>0) =$$
$$= p(1-\Phi(-2)) + (1-p)(1-\Phi(0)) = p\,\Phi(2) + \frac{(1-p)}{2} = \frac{3}{4}$$
$$p = \frac{1}{4(\Phi(2)-\frac{1}{2})} \sim 0.52384$$

(15) $Y = aX + b = a(\sigma Z + m) + b = a\sigma Z + am + b \sim N(am+b, a^2b^2)$

$Z \sim N(0,1)$