1)

(i)

$$\int_0^3 \lambda x^2 dx = \lambda \left[\frac{x^3}{3} \right]_0^3 = \lambda \cdot 9 = 1 \longrightarrow \lambda = \frac{1}{9}$$

(ii)

$$E[X] = \int_{-\infty}^{+\infty} x f(x)\, dx = \int_0^3 x \lambda x^2 = \lambda \int_0^3 x^3 = \lambda \left[\frac{x^4}{4} \right]_0^3 = \frac{81}{4} \lambda = \frac{81}{4} \cdot \frac{1}{9} = \frac{9}{4}$$

(iii)

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 f(x)\, dx = \int_0^3 \lambda x^4 dx = \lambda \left[\frac{x^5}{5} \right]_0^3 = \frac{243}{5} \cdot \frac{1}{9} = \frac{\cancel{243}^{\,27}}{\cancel{45}_{\,5}} = \frac{27}{5}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{27}{5} - \frac{81}{16} = \frac{432-405}{80} = \frac{27}{80}$$

2)

$$f_X(x) = \begin{cases} \frac{1}{4} & -2 \leq x \leq 2 \\ 0 & \text{altrove} \end{cases} \qquad Y = (X+1)^2$$

(i)

$$E(Y) = E[(X+1)^2] = \int_{-2}^{2} (x+1)^2 f_X(x)\, dx = \frac{1}{4} \int_{-2}^{2} (x+1)^2 dx = \frac{1}{4} \left[\frac{(x+1)^3}{3} \right]_{-2}^{2} =$$

$$= \frac{1}{4} \left[9 + \frac{1}{3} \right] = \frac{1}{4} \cdot \frac{28}{3} = \frac{\cancel{28}^{\,7}}{\cancel{12}_{\,3}} = \frac{7}{3}$$

(ii)

$$E[(X+1)^4] = \frac{1}{4} \left[\frac{(x+1)^5}{5} \right]_{-2}^{2} = \frac{1}{4} \left[\frac{243}{5} + \frac{1}{5} \right] = \frac{61}{5}$$

$$Var(Y) = Var((X+1)^2) = E[(X+1)^4] + E[(X+1)^2]^2 = \frac{61}{5} - \frac{49}{9} = \frac{549-245}{45}$$

$$= \frac{304}{45} =$$

3) $X \sim B(5, \frac{1}{3})$ $\quad p(k) = \binom{5}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{n-k}$ $\quad X : \Omega \rightarrow [0,5]$

(i)

$p(0) = \left(\frac{2}{3}\right)^5 = \frac{64}{243}$ $\quad p(1) = \frac{5}{3}\left(\frac{2}{3}\right)^4 = \frac{5}{3}\left(\frac{32}{81}\right) = \frac{160}{243}$

$p(2) = \frac{10}{9}\left(\frac{2}{3}\right)^3 = \frac{10}{9}\left(\frac{8}{27}\right) = \frac{80}{243}$ $\quad p(3) = \frac{10}{27}\left(\frac{2}{3}\right)^2 = \frac{10}{27}\left(\frac{4}{9}\right) = \frac{40}{243}$

$p(4) = \frac{5}{81}\left(\frac{2}{3}\right) = \frac{10}{243}$ $\quad p(5) = \frac{1}{243}$

$$E[X] = \sum_{i=0}^{5} x_i \, p(x_i) = 0 \cdot p(0) + 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3) + 4 \cdot p(4) + 5 \cdot p(5) = \frac{5}{3}$$

(Altro modo) $\rightarrow E[X] = n \cdot p = \frac{5}{3}$

$Var(X) = np(1-p) = 5 \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{10}{9}$

(ii) $X_2 \sim B(5, \frac{1}{3})$ $\quad p(k) = \binom{5}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{n-k}$ $\quad X_2 : \Omega \rightarrow [5,10]$

$$E[X_2] = \sum_{i=5}^{10} x_i \, p(x_i) = 5 \cdot p(0) + 6 \cdot p(1) + 7 \cdot p(2) + 8 \cdot p(3) + 9 \cdot p(4) + 10 \cdot p(5) = \frac{20}{3}$$

(Altro modo) $\rightarrow E[X_2] = E[X+5] = E[X] + 5 = \frac{5}{3} + 5 = \frac{20}{3}$

$Var(X_2) = Var(X+5) = 1^2 \cdot Var(X) = \frac{10}{9}$

(iii)

$X_3 \sim B(5, \frac{1}{3})$ $\quad X_3 : \Omega \rightarrow [10, 20]$

$E[X_3] = E[2X_2] = 2 \cdot E[X_2] = \frac{40}{3}$

$Var(X_3) = Var(2X_2) = 4 \cdot Var(X^2) = \frac{40}{9}$

4) $p(k) = \frac{1}{2^{k-1}} \cdot \frac{1}{2} = \frac{1}{2^k}$ $\quad$ X assume valori in $\{2-k\}$

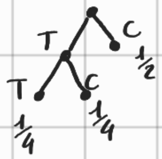
$$E[X] = \sum_{k \geq 1} (2-k) \frac{1}{2^k} = 2 \underbrace{\sum_{k\geq 1} \frac{1}{2^k}}_{1} - \sum_{k \geq 1} k \cdot \frac{1}{2^k} = 2 - 2 = 0$$

5) (i) – (ii)

$P(x_i = C) = \frac{1}{2} + \frac{1}{4} = 3/4$

$P(x_i = T) = \frac{1}{4}$

Esiti del lancio di una moneta



$$Y_i = \begin{cases} 0 & x_i = C \\ 1 & x_i = T \end{cases} \qquad P(Y_i = 0) = 3/4 \quad P(Y_i = 1) = \frac{1}{4} \quad Y_i \sim B(1, \tfrac{1}{4})$$

$$Z = \{\# T\} = Y_1 + Y_2 + Y_3 + Y_4 + Y_5 \sim B(5, 1/4)$$

$$E[Z] = 5 \cdot \frac{1}{4} = 5/4 \qquad Var(Z) = E[Z^2] - E[Z]^2 = n \cdot p \cdot (1-p) =$$
$$= 5 \cdot \frac{1}{4} \cdot \frac{3}{4} = = \frac{15}{16}$$

(iii)

$$3\, P(Z \geq 3) \leq E[Z] = \rightarrow 3P(Z \geq 3) \leq \frac{5}{4} \longrightarrow P(Z \geq 3) \leq \frac{5}{12} < \frac{1}{2}$$
$$\longrightarrow P(Z > 3) = P(Z = 2.5) > \frac{1}{2}$$

6) $Z_1 = (X+1)Y^2 \qquad Z_2 = X^2(Y+1)^2 \qquad Y \sim N(0,9)$

$$E[Z_1] = E[(X+1)Y^2] = E[X+1] \cdot E[Y^2] = (E[X]+1) \cdot E[Y^2] =$$
$$= \left(\frac{2}{9} + 1\right) \cdot \left(Var(Y) + E[Y]^2\right) = \frac{11}{9}\left(9 + 0^2\right) = 11$$

$$E[Z_2] = E[X^2(Y+1)^2] = E[X^2] \cdot E[(Y+1)^2] = (Var(X) + E[X]^2) \cdot E[(Y+1)^2]$$
$$= \left(2 \cdot \frac{1}{9} \cdot \frac{8}{9} + \frac{4}{81}\right) \cdot E[(Y+1)^2] = \frac{20}{81} \cdot \left(E[Y]+1 + Var(Y)\right) =$$

$$= \frac{20}{81} \cdot (0 + 1 + 9) = \frac{200}{81}$$

7) $X \sim B(2, \frac{1}{5})$ $\quad Z = \dfrac{X(Y-1)^2}{X+1}$ $\quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

$$f_Y(y) = \begin{cases} \frac{1}{2} & 0 \leq x \leq 2 \\ 0 & \text{altrove} \end{cases}$$

$$E[(Y-1)^2] = \int_0^2 (y-1)^2 \cdot f_Y(y)\, dy = \frac{1}{2}\int_0^2 y^2 - 2y + 1\, dy =$$

$$= \frac{1}{2}\left(\int_0^2 y^2\, dy - 2\int_0^2 y\, dy + \int_0^2 1\, dy\right) = \frac{1}{2}\left(\left[\frac{y^3}{3}\right]_0^2 - 2\left[\frac{y^2}{2}\right]_0^2 + [y]_0^2\right) =$$

$$= \frac{1}{2}\left(\frac{8}{3} - 4 + 2\right) = \frac{1}{3}$$

$$p(1) = \binom{2}{1}\left(\frac{1}{5}\right)\left(\frac{4}{5}\right) = \frac{16}{25} \qquad p(2) = \frac{1}{25}$$

$$E\left[\frac{X}{X+1}\right] = \sum_{x_i} \frac{x_i}{x_i+1}\, p(x_i) = \sum_{k=0}^{2} \frac{k}{k+1}\, p(k) = \frac{1}{2}\cdot p(1) + \frac{2}{3}\, p(2) =$$

$$= \frac{4}{25} + \frac{2}{75} = \frac{14}{75}$$

$$E[Z] = E\left[\frac{X(Y-1)^2}{X+1}\right] = E\left[\frac{X}{X+1}\right]\cdot E[(Y-1)^2] = \frac{14}{75}\cdot\frac{1}{3} = \frac{14}{225}$$

8) X_1, X_2 $\quad p(k) = e^{-\lambda}\cdot\dfrac{\lambda^k}{k!}$

(i)

$$E[(X_1 - X_2)^2] = 1 \iff 2\lambda = 1 \iff \lambda = \frac{1}{2}$$

$$E[(X_1 - X_2)^2] = E[X_1^2 - 2X_1X_2 + X_2^2] = E[X_1^2] + E[X_2^2] - 2E[X_1]E[X_2] =$$

$$= 2(\lambda + \lambda^2) - 2\lambda^2 = 2\lambda + \cancel{2\lambda^2} - \cancel{2\lambda^2} = 2\lambda$$

(ii)

$$p(k) = e^{-2\lambda} \cdot \frac{2\lambda^k}{k!}$$

$P(X_1 + X_2 \geq 2)$ $\qquad Y = X_1 + X_2$ con parametro 2λ

$$P(X_1 + X_2 \geq 2) = 1 - P(X_1 + X_2 = 0) - P(X_1 + X_2 = 1) = 1 - 2e^{-2\lambda} - e^{-2\lambda}(2\lambda) =$$
$$= 1 - \frac{2}{e} - \frac{1}{e} = 1 - \frac{3}{e}$$

(iii)

$E[g(X_1)]$ $\qquad g(t) = \begin{cases} t & t \leq 2 \\ 2 & t > 2 \end{cases}$ $\qquad$ assume valori in $\{0, 1, 2\}$

$$E[g(X_1)] = \sum_{k=0}^{+\infty} g(k) P(X_1 = k) = g(0) P(X_1 = 0) + g(1) P(X_1 = 1) +$$
$$+ g(2) P(X_1 \geq 2) = \lambda e^{-\lambda} + 2(1 - P(X_1 = 0) - P(X_1 = 1)) =$$
$$= \lambda e^{-\lambda} + 2(1 - e^{-\lambda} - \lambda e^{-\lambda}) = \frac{\lambda}{e^{\lambda}} + \frac{2(e^{\lambda} - 1 - \lambda)}{e^{\lambda}} =$$
$$= \frac{2e^{\lambda} - 2 - \lambda}{e^{\lambda}} = \frac{2e^{\frac{1}{2}} - 2 - \frac{1}{2}}{e^{\frac{1}{2}}} = \frac{4e^{\frac{1}{2}} - 5}{2e^{\frac{1}{2}}} = 2 - \frac{5}{2} e^{-\frac{1}{2}}$$