1) $X_k \sim B(10, \theta)$ 5 $\quad m = 10 \quad n = 5$

$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ m & \theta & n \end{array}$ $\quad x_1 = 6 \quad x_2 = 7 \quad x_3 = 7 \quad x_4 = 8 \quad x_5 = 2$

• <u>Verosomiglianza</u>:

$$L(\theta; x_1, \ldots, x_n) = \theta^{x_1 + \ldots + x_n} (1-\theta)^{m \cdot n - (x_1 + \ldots + x_n)} \prod_{i=1}^{n} \binom{m}{x_i}$$

$$L(\theta; x_1, \ldots, x_n) = \theta^{30} \cdot (1-\theta)^{20} \prod_{i=1}^{5} \binom{10}{x_i}$$

$$L'(\theta) = 30\theta^{29} (1-\theta)^{20} \prod_{i=1}^{5} \binom{10}{x_i} - \theta^{30} 20 (1-\theta)^{19} \prod_{i=1}^{5} \binom{10}{x_i} =$$

$$= \theta^{29} (1-\theta)^{19} \prod \binom{10}{x_i} \left[30(1-\theta) - 20\theta \right] =$$

$$= \theta^{29} (1-\theta)^{19} \prod \binom{10}{x_i} \left[30 - 50\theta \right] = 0 \iff 50\theta = 30 \iff \hat{\theta} = \frac{3}{5}$$

• <u>Momenti</u>:

$$E[X_k] = \underbrace{m \cdot \theta}_{m \cdot p} \Rightarrow m\theta = \sum_{i=1}^{n} \frac{x_i}{n} \Rightarrow \tilde{\theta} = \frac{1}{10} \sum_{i=1}^{5} \frac{x_i}{5} = \frac{3\cancel{0}}{5\cancel{0}} = \frac{3}{5}$$

Segue che $\tilde{\theta} = \hat{\theta} = 3/5$

2) $p(T) = p(C) = 1/2 \qquad X_k \sim B(m, 1/2)$ 5 $\quad m = ? \quad n = 5 \quad \theta = \frac{1}{2}$

$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ m & \theta & n \end{array}$

$x_1 = 692 \quad x_2 = 695 \quad x_3 = 665 \quad x_4 = 674 \quad x_5 = 719$

• <u>Verosomiglianza</u>:

$$L(\theta; x_1, \ldots, x_n) = \theta^{x_1 + \ldots + x_n} (1-p)^{m \cdot n - (x_1 + \ldots + x_n)} \prod_{i=1}^{5} \binom{m}{x_i}$$

$$L(\theta; x_1, \ldots, x_n) = \left(\frac{1}{2}\right)^{3445} \left(\frac{1}{2}\right)^{5m - 3445} \prod_{i=1}^{5} \binom{m}{x_i}$$ Non si può derivare

$X_k = Y_1 + \dots + Y_n$ con $Y_i \sim B(1, 1/2)$

$$Z = \frac{Y_1 + \dots + Y_n - n\mu}{\sqrt{n}\,\sigma} \sim N(0,1) \implies X_k = \frac{\sigma}{\sqrt{n}} Y_n + \mu$$

$X_k \sim N(n\mu, n\sigma^2)$ con $\mu = \frac{1}{2}$ e $\sigma^2 = \frac{1}{4}$ $\qquad m = \mu$

$$L(\theta; x_1, \dots, x_5) = \prod_{i=1}^{5} \frac{1}{\sqrt{2\pi \frac{m}{4}}} e^{-\frac{(x_i - \frac{m}{2})^2}{\frac{2m}{4}}} = \left(\frac{2}{\pi}\right)^{\frac{5}{2}} \cdot \theta^{-\frac{5}{2}} \cdot e^{-\frac{2}{\theta}[(x_1 - \frac{\theta}{2})^2 + \dots + (x_5 - \frac{\theta}{2})^2]}$$

$$\log(L) = \frac{5}{2}\log\left(\frac{2}{\pi}\right) - \frac{5}{2}\log(\theta) - \frac{2}{\theta}[(x_1 - \tfrac{\theta}{2})^2 + \dots + (x_5 - \tfrac{\theta}{2})^2]$$

$$\frac{d}{d\theta}\log(L) = -\frac{5}{2\theta} + \frac{2}{\theta^2}[(x_1 - \tfrac{\theta}{2})^2 + \dots + (x_5 - \tfrac{\theta}{2})^2] + \frac{2}{\theta}[(x_1 - \tfrac{\theta}{2}) + \dots + (x_5 - \tfrac{\theta}{2})] =$$

$= \dots$

• Momenti :

$$E[X_k] = \frac{m}{2} \implies \frac{\tilde{m}}{2} = \sum_{i=1}^{5} \frac{x_i}{5} \implies \tilde{m} = \frac{2}{5}(692 + 695 + 665 + 674 + 719) = 1378$$

3) $n = 5$ $\quad x_1 = 0.23 \quad x_2 = 0.46 \quad x_3 = 0.78 \quad x_4 = 0.63 \quad x_5 = 0.90$

• Verosomiglianza :

$$L(\theta; x_1, \dots, x_5) = \prod_{i=1}^{5} f_\theta(x_i) = \prod_{i=1}^{5} \theta x_i^{\theta-1} = \theta^5 (x_1^{\theta-1} \cdot x_2^{\theta-1} \cdot x_3^{\theta-1} \cdot x_4^{\theta-1} \cdot x_5^{\theta-1}) =$$

$$= \theta^5 (x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5)^{\theta-1}$$

$$L'(\theta) = 5\theta^4 (x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5)^{\theta-1} + \theta^5 \ln(x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5)(x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5)^{\theta-1}$$

$$= \theta^4 (x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5)^{\theta-1} [5 + \theta \ln(x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5)] = 0 \iff$$

$$\iff \hat{\theta} = -\frac{5}{\ln(x_1 \cdot \ldots \cdot x_5)} \sim 1.633$$

· *Momenti*:

$$E[X] = \int_0^1 x f(x) = \int_0^1 x\, \theta x^{\theta-1}\, dx = \theta \int_0^1 x^\theta\, dx = \theta \left[\frac{x^{\theta+1}}{\theta+1} \right]_0^1 = \frac{\theta}{\theta+1}$$

$$\frac{\theta}{\theta+1} = \sum_{i=1}^{5} \frac{x_i}{5} = \frac{3}{5} \Rightarrow \tilde{\theta} = \frac{3}{5}(\tilde{\theta}+1) \Rightarrow \tilde{\theta} - \frac{3}{5}\tilde{\theta} = \frac{3}{5}$$

$$\tilde{\theta} = \frac{3}{2} = 1.5$$

4) $X \sim N(\theta, 1)$ $\qquad \mu = m = \theta \qquad \sigma^2 = 1$

$x_1 = 10 \quad x_2 = 15 \quad x_3 = 20 \quad x_4 = 17 \quad x_5 = 13 \qquad \sum_{i=1}^{5} x_i = 75$

· *Verosomiglianza*:

$$L(\theta; x_1, \dots, x_5) = \prod_{i=1}^{5} f_\theta(x_i) = \prod_{i=1}^{5} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i-\theta)^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[(x_1-\theta)^2 + \dots + (x_5-\theta)^2]}$$

$$\log(L) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}[(x_1-\theta)^2 + \dots + (x_5-\theta)^2]$$

$$\log'(L) = -\frac{1}{2}[(-2)(x_1-\theta) + \dots + (-2)(x_5-\theta)] = (x_1-\theta) + \dots + (x_5-\theta) =$$
$$= x_1 + x_2 + x_3 + x_4 + x_5 - 5\theta = 75 - 5\theta = 0 \Leftrightarrow \hat{\theta} = \frac{75}{5} = 15$$

· *Momenti*:

$$E[X] = \theta \qquad \theta = \sum_{i=1}^{5} \frac{x_i}{5} \Rightarrow \tilde{\theta} = \frac{75}{5} = 15 \qquad \hat{\theta} = \tilde{\theta}$$

5) (i)

$$\frac{d}{dx} F_\theta(x) = \begin{cases} 0 \\ \frac{1}{1+\theta}(2x+\theta) \\ 0 \end{cases} \longrightarrow \begin{cases} 2x+\theta \geq 0 \\ 1+\theta > 0 \end{cases} \vee \begin{cases} 2x+\theta \leq 0 \\ 1+\theta < 0 \end{cases}$$

$$\begin{cases} \theta \geq -2x \\ \theta > -1 \end{cases} \vee \begin{cases} \theta \leq -2x \\ \theta < -1 \end{cases} \qquad x \in [0,1]$$

$$\theta \geq 0 \quad \vee \quad \theta \leq -2$$

$$\lim_{x \to 0^+} F(x) = 0$$

$$\lim_{x \to 1^-} F(X) = 1$$

(ii)

$x_1 = 0.35 \quad x_2 = 0.40 \quad x_3 = 0.31 \quad x_4 = 0.85 \quad x_5 = 0.59 \quad x_6 = 0.60 \qquad n = 6$

$$E[X_k] = \int_0^1 x f(x)\,dx = \int_0^1 x \, \frac{(2x+\theta)}{1+\theta}\,dx = \frac{1}{1+\theta}\int_0^1 2x^2 + \theta x \,dx =$$

$$= \frac{1}{\theta+1}\left[2\int_0^1 x^2 + \theta\int_0^1 x\right] = \frac{1}{\theta+1}\left[2\left[\frac{x^3}{3}\right]_0^1 + \theta\left[\frac{x^2}{2}\right]_0^1\right] =$$

$$= \frac{1}{\theta+1}\left[\frac{2}{3} + \frac{\theta}{2}\right] = \frac{4+3\theta}{6(\theta+1)}$$

$$E[X_k] = \sum_{i=1}^{6} \frac{x_i}{6} \longrightarrow \frac{4+3\theta}{6(\theta+1)} = \frac{3.1}{6} \longrightarrow \cancel{6} \cdot \frac{4+3\theta}{\cancel{6}(\theta+1)} = 3.1 \longrightarrow$$

$$\longrightarrow 4+3\theta = 3.1\,\theta + 3.1 \longrightarrow 0.1\,\theta = 0.9 \longrightarrow \tilde{\theta} = 9$$

6) $x_1 = 4.2 \quad x_2 = 6.3 \quad x_3 = 7.1 \quad x_4 = 5.8 \quad x_5 = 8.3 \qquad \sum_{i=1}^{5} x_i = 31.7$

- <u>Verosomiglianza</u>:

$$L(\theta; x_1, \dots, x_5) = \prod_{i=1}^{5} f_\theta(x_i) = \begin{cases} e^{-[x_1 + \dots + x_5 - 5\theta]} & x_i \geq \theta \quad \forall i \\ 0 & \text{altrimenti} \end{cases}$$

Quindi $\theta \leq \min\{x_i\} = \xi$ e $\hat{\theta} = \xi$ $\quad \hat{\theta} = 4.2$

• **Momenti**:

$$E[X_k] = \int_{\theta}^{+\infty} x\, e^{-(x-\theta)}\, dx \overset{\text{Sia } y = x-\theta}{=} \int_0^{+\infty} (y+\theta)\, e^{-y}\, dy = \int_0^{+\infty} y\, e^{-y}\, dy + \theta \int e^{-y}\, dy =$$

$$= \left[-y e^{-y}\right]_0^{+\infty} - \int_0^{+\infty} -e^{-y}\, dy + \theta = 1 + \theta \rightarrow 1 + \theta = \frac{31.7}{5} \rightarrow \tilde{\theta} = 5.34$$