1) $G_X(t) = E[e^{tX}] = \sum_{x_i} e^{tx_i} p(x_i) = (1-p) + e^t p \qquad p(k) = \binom{n}{k}(1-p)^{n-k} p^k$

$$E[X^k] = \left(\frac{d^k}{dt^k} G\right)(0) = \left(\frac{d^k}{dt^k} e^t\right)(0) = p \quad \forall k \geq 1$$

2) $$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{altrove} \end{cases}$$

$$G_X(t) = \int_0^1 e^{tx} f_X(x)\, dx = \int_0^1 e^{tx}\, dx = \begin{cases} \left[\frac{1}{t} e^{tx}\right]_0^1 = \frac{e^t - 1}{t} & t \neq 0 \\ \int_0^1 1\, dx = 1 & t = 0 \end{cases}$$

$e^t = 1 + t + \frac{t^2}{2}$

$$E[X] = G_X'(0) = \lim_{t\to 0} \frac{G_X(t) - G_X(0)}{t} = \frac{\frac{e^t - 1}{t} - 1}{t} = \frac{e^t - 1 - t}{t^2} = \frac{1}{2}$$

$$G_X'(t) = \begin{cases} \frac{1}{2} & t = 0 \to G_X'(0) = E[X] \\ \frac{e^t t - e^t + 1}{t^2} \end{cases}$$

$$E[X^2] = G_X''(t) = \lim_{t\to 0} \frac{G_X'(t) - G_X'(0)}{t} = \frac{\frac{e^t t - e^t + 1}{t} - \frac{1}{2}}{t} =$$

3) $X_k = \frac{k+1}{k} Y + 1 \qquad \bar{X} = \frac{X_1 + \dots X_n}{n} \qquad X_k \sim N\left(1, \left(\frac{k+1}{k}\right)^2\right)$

$X_1 = 2Y + 1 \qquad E[X_1] = E[2Y+1] = 2\underbrace{E[Y]}_{0} + 1 = 1$

$\vdots$

$X_n = \frac{n+1}{n} Y + 1 \qquad E[X_n] = E\left[\left(\frac{n+1}{n}\right)Y + 1\right] = \left(\frac{n+1}{n}\right)E[Y] + 1 = 1$

$E[\bar{X}_n] = \frac{1}{n} \sum_{k=1}^{+\infty} E[X_k] = 1 \qquad \lim_{n\to+\infty} \{|\bar{X}_n - E[\bar{X}_n]| > \varepsilon\} \to 0$

$P\{\bar{X}_n \in (\frac{1}{4}, \frac{1}{2})\} \to 0$ perché $E[\bar{X}_n] = 1 \notin (\frac{1}{4}, \frac{1}{2})$

(ii)

$$X_k = \frac{k+1}{k} Y + (-1)^k \qquad \bar{X} = \frac{X_1 + \dots X_n}{n}$$

$$E[\bar{X}_k] = \begin{cases} -1 & k \text{ dispari} \\ 0 & k \text{ pari} \end{cases} \qquad \lim_{n \to +\infty} \{|\bar{X}_n - E[\bar{X}_n]| > \varepsilon\} \to 0$$

$P\{\bar{X}_n \in (\frac{1}{4}, \frac{1}{2}]\} \to 0$ perché $-1 \notin (\frac{1}{4}, \frac{1}{2})$ e $0 \notin (\frac{1}{4}, \frac{1}{2})$

4) $X_k : \Omega \to \{0, 1\}$ $\quad P(X_k = 0) = P(X_k = 1) = \frac{1}{2} \quad X_k \sim B(1, \frac{1}{2})$

$\mu = E[X_k] = n \cdot p = \frac{1}{2} \qquad \sigma^2 = Var(X_k) = n \cdot p(1-p) = \frac{1}{4}$

$$\bar{X}_{100} = \frac{X_1 + \dots + X_{100}}{100} \qquad Y_{100} = \frac{X_1 + \dots + X_{100} - 100\mu}{\sqrt{100\sigma^2}}$$

$$\bar{X}_{100} = \frac{\sigma}{\sqrt{100}} Y_{100} + \mu$$

(i)

$$P(X_1 + \dots + X_{100} < 70) = P\left(Y_{100} < \frac{70 - 100 \cdot \frac{1}{2}}{10 \cdot \frac{1}{2}}\right) = P(Y_{100} < 4) \sim$$

$$\sim P(Z < 4) = \Phi(4) \sim 0.9997$$

(ii)

$$P\left(Y_{100} \geq \frac{65 - 50}{5}\right) = P(Z \geq 3) = 1 - \Phi(3) \sim 0.00135$$

(iii)

$$P\left(\frac{40 - 50}{5} < Y_{100} < \frac{55 - 50}{5}\right) = P(-2 < Z < 1) = \Phi(1) - \Phi(-2) =$$

$$= \Phi(1) - (1 - \Phi(2)) \sim 0.81859$$

5) X_k con $E[X_k] = 1200$ $\sigma = 200$

$X_1 + \dots X_{100} \longrightarrow Y_{100} = \dfrac{X_1 + \dots + X_{100} - 100\mu}{\sqrt{100} \cdot \sigma}$

(i)

$P(X_1 + \dots + X_{100} \geq 122000) = P\left(\dfrac{X_1 + \dots + X_{100} - 120000}{2000} \geq \dfrac{122000 - 120000}{2000}\right)$

$= P(Y_1 \geq 1) = 1 - \Phi(1) \sim 0.15866$

(ii)

Cerco n t.c. $P(1150 < \overline{X}_n < 1250) \geq 0.999$

$\overline{X}_n = \dfrac{\sigma}{\sqrt{n}} Y_n + 1200$

$P(1150 < \overline{X}_n < 1250) = P(1150 < \frac{200}{\sqrt{n}} Y_n + 1200 < 1250) =$
$= P(-\frac{50}{200}\sqrt{n} < Y_n < \frac{50}{200}\sqrt{n}) \sim P(-\frac{1}{4}\sqrt{n} < Z < \frac{1}{4}\sqrt{n}) =$
$= \Phi(\frac{1}{4}\sqrt{n}) - \Phi(-\frac{1}{4}\sqrt{n}) = 2\Phi(\frac{1}{4}\sqrt{n}) - 1 = 0.999 \iff$
$\iff \Phi(\frac{1}{4}\sqrt{n}) = \frac{1.999}{2} = 0.995 \iff \frac{\sqrt{n}}{4} = 3.29 \iff n \sim 174$

6) $\mu = 21$ $\sigma^2 = 7$ $\overline{X}_n = \dfrac{\sqrt{7}}{\sqrt{n}} Y_n + 21$

Cerco n t.c. $P(19 < X_n < 23) \geq 0.99$

$P(19 < X_n < 23) = P(19 < \frac{\sqrt{7}}{\sqrt{n}} Y_n + 21 < 23) = P(-\frac{2}{\sqrt{7}}\sqrt{n} < Y_n < \frac{2}{\sqrt{7}}\sqrt{n}) =$
$= 2\Phi(\frac{2}{\sqrt{7}}\sqrt{n}) - 1 = 0.99 \iff \Phi(\frac{2}{\sqrt{7}}\sqrt{n}) = \frac{1.99}{2} = 0.995 \iff$
$\iff \frac{2}{\sqrt{7}}\sqrt{n} \geq 2.58 \iff n \geq \left(\frac{\sqrt{7} \cdot 2.58}{2}\right)^2 \sim 12$

7) $X \sim N(100, 1)$ $\quad m = \mu = 100 \quad \sigma^2 = \sigma = 1$

(i)

$$\bar{X}_{1000} = \frac{1}{\sqrt{1000}} Y_{1000} + 100 \qquad X = Z + 100 \quad \text{con } Z \sim N(0,1)$$

$$P(X_k > 100.5) = P(X > 100.5) = P(Z > 0.5) = 1 - \Phi(0.5) \sim 0.30854$$

(ii)

$Y_k \sim B(1, p)$ che vale 0 se $X_k \leq 100.5$ e 1 se $X_k > 100.5$ e

$p = P(Y_k = 1) = P(X_k > 100.5) \sim 0.30854$

$E[Y_k] = p \qquad Var(Y_k) = p(1-p)$

$$P(Y_1 + \ldots + Y_{1000} \leq 300) = P\left(\frac{Y_1 + \ldots + Y_{1000} - 1000 \cdot p}{\sqrt{1000\, p(1-p)}} \leq \frac{300 - 1000p}{\sqrt{1000\, p(1-p)}}\right) \sim$$

$$\sim P\left(Z \leq \frac{300 - 1000p}{\sqrt{1000\, p(1-p)}}\right) = \Phi\left(\frac{300 - 1000p}{\sqrt{1000\, p(1-p)}}\right) = \Phi\left(\frac{300 - 308.54}{\sqrt{1000 \cdot 0.21334}}\right) = \Phi\left(-\frac{8.54}{14.60}\right)$$

$$= \Phi(-0.58468) = 1 - \Phi(0.58468) = 1 - 0.72 \sim 0.28$$