Dimostrazione del limite superiore dell'entropia congiunta

$H(X,Y) \leq H(X)+H(Y)$, e l'uguaglianza si ha quando X e Y sono indipendenti.

$$\begin{aligned}
H(X) &= -\sum_{i=1}^{h} p(x_i)\log p(x_i) = -\sum_{i=1}^{h}\left(\sum_{j=1}^{k} p(x_i,y_j)\right)\log p(x_i) \\
H(Y) &= -\sum_{j=1}^{k} p(y_j)\log p(y_j) = -\sum_{j=1}^{k}\left(\sum_{i=1}^{h} p(x_i,y_j)\right)\log p(y_j) \\
H(X)+H(Y) &= -\sum_{j=1}^{k}\sum_{i=1}^{h} p(x_i,y_j)\left(\log p(x_i)+\log p(y_j)\right) \\
&= -\sum_{j=1}^{k}\sum_{i=1}^{h} p(x_i,y_j)\log(p(x_i)p(y_j))
\end{aligned}$$

Ponendo $q(x_i,y_j) = p(x_i)p(y_j)$, $(\sum_i \sum_j q(x_i,y_j) = 1)$ applichiamo il lemma del logaritmo:

$$\underbrace{-\sum_{i=1}^{h}\sum_{j=1}^{k} p(x_i,y_j)\log p(x_i,y_j)}_{H(X,Y)} \leq \underbrace{-\sum_{i=1}^{h}\sum_{j=1}^{k} p(x_i,y_j)\log q(x_i,y_j)}_{H(X)+H(Y)}.$$

Se X e Y sono indipendenti, $p(x_i,y_j) = q(x_i,y_j)$.